Diffusion equations for a Markovian jumping process

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We consider a Markovian jumping process which is defined in terms of the jump-size distribution and the waiting-time distribution with a position-dependent frequency, in the diffusion limit. We assume the power-law form for the frequency. For small steps, we derive the Fokker-Planck equation and show the presence of the normal diffusion, subdiffusion, and superdiffusion. For the Lévy distribution of the step size, we construct a fractional equation, which possesses a variable coefficient, and solve it in the diffusion limit. Then we calculate fractional moments and define the fractional diffusion coefficient as a natural extension to the cases with the divergent variance. We also solve the master equation numerically and demonstrate that there are deviations from the Lévy stable distribution for large wave numbers.

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I. INTRODUCTION

The diffusion is called anomalous if the mean squared displacement of the Brownian particle does not rise with time linearly, as is the case for the usual Brownian motion, but slower (subdiffusion) or faster (superdiffusion). There are many examples of physical systems which exhibit such anomalous transport; they involve complex systems, disordered systems, semiconductors, polymers, glasses, turbulent plasma, and many others (for a review, see Refs. [1,2]). Physical situations in which the deviation of the linear dependence is expected comprise: a porous and inhomogeneous environment with traps and barriers, a coupling to a fractal heat bath via a random matrix interaction [3], long-range and/or long-time correlations. Also the transport phenomena in dynamical Hamiltonian systems often exhibit anomalous behavior because the corresponding phase space can contain some regular structures which act as dynamical traps [4,5]. A trajectory sticks to such self-similar hierarchy of islands in a chaotic environment and abide on regular paths for a long time. The emergence of the anomalous diffusion means that the traditional approach, involving the standard Fokker-Planck equation (FPE) with the diffusion constant, is no longer valid. Usually, the anomalous diffusion is attributed to memory effects, which are especially pronounced in non-Markovian descriptions, e.g., by means of the generalized Langevin equation [6] and the continuous-time random walk models (CTRW) [1,7]. In the diffusive limit, those approaches resolve themselves to the fractional equations formalism [1,8,9] which contain the Riemann-Liouville operator as the fractional time derivative; the step-size distribution is the Gaussian. As a direct consequence of the nonlocality in time and therefore of the lack of Markovian property, the uncoupled CTRW predicts subdiffusive solutions in this case [10,11]. However, the anomalous behavior is possible also for the Markovian processes; they are described by FPE with the position-dependent diffusion term and physically can correspond, e.g., to the diffusion on fractal objects [12] (this problem has been further developed and reformulated in terms of the fractional equations in Refs. [13–15]) and to the Langevin equation with the multiplicative noise [16]. In fact, the enhanced diffusion can emerge also for the finite mean waiting time if the memory term in the CTRW is coupled [17]. In this paper we discuss an example of the Markovian process which exhibits all kinds of diffusion.

A more general approach considers the Lévy distribution which is defined, in its symmetric version, via the characteristic function $\phi(k)=\exp(-\alpha|k|^{\mu})$, where $0<\mu\leq 2$. If $\mu\neq 2$ the second moment is infinite and the generalized central limit theorem must be applied. Physically, Lévy processes can reflect self-similarity: they are the stable solutions of the renormalization group method and are invariant under the scaling of position and time. In contrast to the Gaussian distribution, they include large fluctuations. The Lévy distributions are present in many problems connected with various branches of science, including not only physics but also biology, economy, financial market research, etc. Recent studies of systems which are characterized by the stable distributions include stochastic equations with potentials [18] and a mean first passage time analysis [19]. The diffusion equation for the Lévy process involves a fractional derivative over the process value [5,20–23] and for $\mu=2$ it resolves itself to the ordinary FPE. Since the variance diverges for $\mu<2$ and the traditional description of the diffusion process is no longer valid, one has to introduce new concepts, e.g., to study fractional moments [24] or to restrict the integration in averaging to a finite box which scales with time in a prescribed way [25,26].

Problems related to generalized diffusion equations, which contain either anomalous behavior of the variance or infinite fluctuations, are the subject of the present paper. We deal with a jumping process which is Markovian, defined in terms of a jump-size distribution $Q(x)$ and the waiting time distribution $P_P(\tau)$. A peculiarity of the distribution $P_P(\tau)$ consists in a position-dependent jumping rate. The process is defined in Sec. II. In Sec. III we derive the FPE as a small step limit of the master equation. Section IV is devoted to the fractional diffusion equation which is an approximation of the master equation in the diffusion limit for Q given by the Lévy distribution; we solve this equation, discuss its diffusion properties, and compare with numerical solution of the master equation. The results of our analysis are discussed in Sec. V.

II. DEFINITION OF THE PROCESS

The jumping process we are to deal with in this paper is a stepwise constant stochastic process $x(t)$ defined in terms of two probability distributions [27]. The waiting time density distribution determines the time intervals τ_i between consecutive jumps and it is assumed in the Poissonian form:

$$P_P(\tau) = \nu(x)e^{-\nu(x)\tau}, \quad (1)$$

where the jumping rate $\nu(x)$ depends on the process value (the position). The size of the jumps is determined by a given normalized distribution $Q(r=x-x')$. Then the infinitesimal transition probability can be easily constructed and the master equation derived. It is of the form

$$\frac{\partial}{\partial t}p(x,t) = -\nu(x)p(x,t) + \int Q(x-x')\nu(x')p(x',t)dx'. \quad (2)$$

Because of the variable jumping rate $\nu(x)$, the above process can be regarded as a version of the kangaroo process [28,29]. The difference consists in the jump-size dependence of Q—in the kangaroo process Q depends only on the current position. Due to that property, the Eq. (2) can describe transport phenomena [27]. On the other hand, since the waiting time distribution depends on the position, the process constitutes a special case of the coupled CTRW. Taking into account the position dependence of the jumping rate is an important generalization of traditional random walk approaches. Such dependence is expected in many phenomena in which inhomogeneity of the environment cannot be neglected [30].

The stationary solution of Eq. (2) can be easily obtained: $P(x)=1/\nu(x)$. The normalization condition imposes restrictions on the function $\nu(x)$: it must rise sufficiently fast in the infinity and sufficiently slow near zero. If, in turn, $1/\nu(x)$ has poles at some x, the stationary solution also exists and it is in the form of a combination of the delta functions. In the other cases $P(x)$ does not exist. A special version of the stationary process, defined on the circle, exhibits long-time correlations and can serve as a model of the $1/f$ noise [31,32].

The general, time-dependent solution of Eq. (2) can be obtained by using the Laplace transforms. The formal expression for the Laplace transform of $p(x,t)$ is the following [27]:

$$\tilde{p}(x,s) = \frac{p_0(x)}{s+\nu(x)} + \frac{1}{s+\nu(x)}\int \frac{\nu(x_0)p_0(x_0)Q(x-x_0)}{s+\nu(x_0)}dx_0 + \frac{1}{s+\nu(x)}\sum_{k=2}^{\infty}\int \frac{\nu(x_0)p_0(x_0)}{s+\nu(x_0)}Q(x-x_{k-1}) \times \left[\prod_{i=2}^{k}\frac{\nu(x_{i-1})Q(x_{i-1}-x_{i-2})}{s+\nu(x_{i-1})}dx_{i-1}\right]dx_0, \quad (3)$$

where $p_0(x)$ stands for the initial condition.

By multiplying Eq. (2) by x^2 and by integrating over x, one can obtain the equation which governs the time evolution of the variance. Assuming that Q has the vanishing mean and finite second moment, we yield the following result:

$$\frac{\partial\langle x^2\rangle_p}{\partial t} = \langle x^2\rangle_Q\langle\nu\rangle_p. \quad (4)$$

The simple case $\nu(x)=\text{const}$ can be solved completely and Eq. (4) predicts the normal diffusion. However, if $\langle x\rangle_Q$ does not vanish, the diffusion becomes ballistic [27]. For Q with divergent second moment, in turn, $\langle x^2\rangle_p$ is infinite.

III. SMALL JUMPS: THE FOKKER-PLANCK EQUATION

Expression (3) is difficult to handle and one has to resort to approximations. In the limit of small jumps, the process becomes continuous both in space and time and FPE may be a candidate for such approximation. In order to construct it, we apply the Kramers-Moyal expansion [33] of the master equation. In that method, one changes the integration variable in Eq. (2) by introducing the step size $r=x-x'$ and, after the expansion of the function $p(x-r,t)\nu(x-r)$ around $r=x$, one obtains the master equation in a form of the following series:

$$\frac{\partial p(x,t)}{\partial t} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\langle r^n\rangle_Q\left(\frac{\partial}{\partial x}\right)^n[p(x,t)\cdot\nu(x)] \quad (5)$$

which is still exact. The approximation consists in neglecting all terms of the order higher than 2. Obviously, all moments of Q must be finite. Finally, we obtain the FPE:

$$\frac{\partial p(x,t)}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2[\nu(x)p(x,t)]}{\partial x^2}, \quad (6)$$

where $\sigma^2=\langle r^2\rangle_Q$. Therefore the jumping rate may be interpreted as the position-dependent diffusion coefficient. The approximation is valid if the jumps are small and the function $\nu(x)$ is smooth [34]:

$$Q(r) \approx 0 \quad \text{for } r > \delta,$$

$$\nu(x+\Delta x) \approx \nu(x) \quad \text{for } \Delta x < \delta, \quad (7)$$

where δ is a small parameter.

For $\nu(x)=\text{const}$, Eq. (6) takes the form of the ordinary diffusion equation which describes the Wiener process and is characterized by the normal diffusion. An interesting case is the power-law form of the jumping frequency:

$$\nu(x) = \gamma|x|^{-\theta}, \quad (8)$$

where θ is a constant parameter and γ ensures proper units; in the following we assume $\gamma=1$. The above expression for the jumping frequency will be applied in this paper. The diffusion coefficient in the form (8) has been used to describe the diffusion on fractal objects [12], the transport of fast electrons in a hot plasma [35], and turbulent two-particle diffusion [36]. The FPE with that diffusion coefficient has been analyzed from the point of view of the first passage time in Ref. [37]. The FPE (6) with ν given by Eq. (8) can be solved exactly [38]:

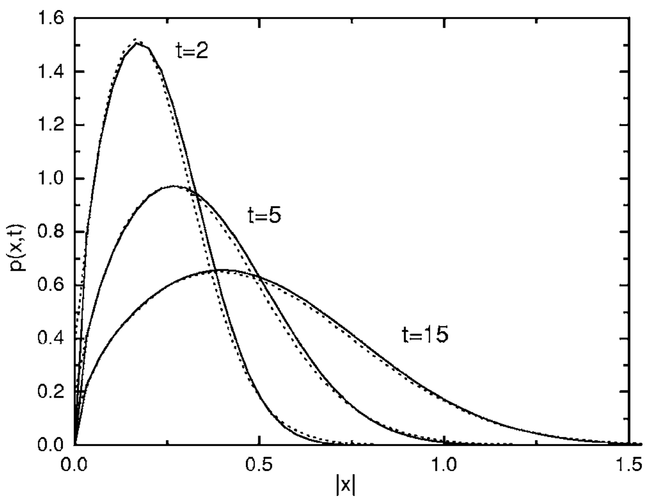

FIG. 1. Time evolution of the FPE solutions $p(x,t)$ calculated from Eq. (9) (solid line) and solutions of master equation (2) for the Gaussian form of the distribution Q (dots). Parameters are $\theta=0.5$ and $\sigma=0.1$.

$$p(x,t) = C_\theta \frac{|x|^\theta \exp(-2|x|^{2+\theta}/\sigma^2(2+\theta)^2 t)}{(\sigma^2 t/2)^{(1+\theta)/(2+\theta)}}, \quad (9)$$

where $C_\theta=\frac{1}{2\Gamma[(1+\theta)/(2+\theta)]}|2+\theta|^{\theta/(2+\theta)}$. Some values of θ must be excluded. Since for $\theta<-3$ Eq. (2) has the stationary solution (in the form of the delta function), the approximation by the FPE is not valid; the diffusion in this case is local. Moreover, for $\theta\in(-2,-1)$ the distribution (9) is not normalized. Therefore we have either $\theta\in(-3,-2)$ or $\theta>-1$.

Figure 1 presents the comparison of the FPE solutions $p(x,t)$, calculated according to Eq. (9), with the solutions of master equation (2), obtained from Monte Carlo simulations of the stochastic trajectories. The agreement is good already for $\sigma=0.1$.

The diffusion properties of the system follow directly from solution (9). The mean squared displacement for $\theta>-1$ is given by the power-law function of time by the following formula:

$$\langle x^2(t)\rangle = \frac{\Gamma[(3+\theta)/(2+\theta)]}{\Gamma[(1+\theta)/(2+\theta)]}\left[\frac{\sigma^2}{2}(2+\theta)^2 t\right]^{2/(2+\theta)} \quad (10)$$

and the diffusion coefficient is

$$\mathcal{D} = \frac{\langle x^2\rangle}{2t} \sim t^{-\theta/(2+\theta)} \quad (t\to\infty). \quad (11)$$

Then we can distinguish three cases. For $\theta\in(-1,0)$, $\mathcal{D}$ is infinite and the superdiffusion emerges. For $\theta>0$, $\mathcal{D}=0$ and we get the subdiffusion. Finally, the normal diffusion takes place for $\theta=0$. Therefore the jumping process involves all kinds of diffusion.

The observed anomalous behavior can be understood if we consider the average waiting time interval $\bar{\tau}=\langle\tau\rangle_{P_P}=1/\nu(x)$. The time dependence of that quantity can be estimated by the average $\langle\bar{\tau}\rangle_{p(x,t)}=\int\bar{\tau}(x)p(x,t)dx\sim t^{\theta/2+\theta}$. Since $\langle\bar{\tau}\rangle_{p(x,t)}$ rises with time for $\theta>0$, the waiting times becomes larger and then the diffusion weakens, compared to the normal case, in accordance with Eq. (10). On the other hand, $\langle\bar{\tau}\rangle_{p(x,t)}$ diminishes with time for $\theta\in(-1,0)$ which results in the enhanced diffusion. Since the distribution $p(x,t)$ widens with time, the large x values are decisive for evaluation of the average $\langle\bar{\tau}\rangle_{p(x,t)}$ in the diffusive limit.

The above observations help us to predict the diffusive properties of the processes defined by frequencies $\nu(x)$ other than the algebraic dependence (8), for which the FPE cannot be solved exactly. For example, if $\nu(x)=\exp(-c|x|)$, $c>0$, the tails fall faster than any power law; we expect especially strong trapping of the stochastic trajectories and the diffusion must be very weak. Indeed, the numerical estimation of the variance $\langle x^2(t)\rangle$ produces the result $\langle x^2(t)\rangle\sim t^{0.009}$.

IV. FRACTIONAL DIFFUSION EQUATION

Let us now assume the distribution Q in the form of the symmetric Lévy distribution, defined by its Fourier transform:

$$\tilde{Q}(k) = \exp(-K^\mu|k|^\mu) \quad (K>0), \quad (12)$$

where $1<\mu<2$. In contrast to the Gaussian distribution, $Q(x)$, corresponding to Eq. (12), has algebraic tails, $Q(x)\sim|x|^{-1-\mu}(|x|\to\infty)$, which makes long jumps very probable. In the diffusion limit $k\to0$, Eq. (12) is given by $\tilde{Q}(k)\approx1-K^\mu|k|^\mu$. We wish to derive an equation which could serve as an approximation to master equation (2) in the diffusion limit. We take the Fourier transform of Eq. (2) and insert $\tilde{Q}(k)$ in the above form, that yields

$$\frac{\partial\tilde{p}(k,t)}{\partial t} = -K^\mu|k|^\mu\mathcal{F}[\nu(x)p(x,t)]. \quad (13)$$

Equation (13) can be formally inverted by expressing the invert Fourier transform by a fractional Weyl-Riesz operator: $\mathcal{F}^{-1}(-|k|^\mu)=\frac{\partial^\mu}{\partial|x|^\mu}$ [39]. The resulting equation is the following:

$$\frac{\partial p(x,t)}{\partial t} = K^\mu\frac{\partial^\mu[\nu(x)p(x,t)]}{\partial|x|^\mu}. \quad (14)$$

Technically, the presence of the x-dependent term under the fractional derivative poses the main difficulty. Nevertheless, in many physical situations the variability of the coefficient in the fractional equation is important. It is the case, for example, in a consistent description of the Lévy flights in complex systems, involving an external periodic potential [40]. Recently, properties of human travels have been studied in terms of the stochastic fractional equations by analyzing the circulation of bank notes in the United States [41]. The lack of expected relaxation of probability distributions to the equilibrium can be caused by the spatial inhomogeneities of the system and then it could be explained by introducing the variable diffusion coefficient.

Our aim is to solve Eq. (14) for ν given by Eq. (8), where $\theta>-1$, with the initial condition $p(x,0)=\delta(x)$.

A. The case $\theta=0$

Equation (13) for this case can be easily solved:

$$\tilde{p}(k,t)=\exp(-K^\mu t|k|^\mu). \tag{15}$$

The above expression is the characteristic function of the Lévy process [5] and its inversion produces the symmetric Lévy distribution. To handle those distributions, it is convenient to apply Fox functions. In fact, the Lévy distribution in its most general form can be expressed as a Fox function [42]. That formalism is useful in describing stable processes because it reflects scaling properties of the underlying phenomena. The definition and some properties of the Fox functions are presented in the Appendix.

The solution of Eq. (14) can then be expressed in the following way [1,43]:

$$p(x,t)=\frac{1}{\mu|x|}H_{2,2}^{1,1}\left[\frac{|x|}{(K^\mu t)^{1/\mu}}\middle|\begin{matrix}(1,1/\mu),(1,1/2)\\(1,1),(1,1/2)\end{matrix}\right]. \tag{16}$$

The asymptotics $|x|\to\infty$ results from the expansion (A3): $p(x,t)\sim t/|x|^{1+\mu}$ and implies that the variance, as well as all fractional moments of the order of μ or higher, diverge.

B. The general case

Equation (14), as an approximation to the master equation in the diffusion limit, has been obtained from Eq. (2) by dropping all terms of the order higher than $|k|^\mu$ in the expansion of $Q(k)$. Consequently, solving Eq. (14), we can neglect those terms without introducing any additional idealization. We look for a solution in the form of the Fox function

$$p(x,t)=NaH_{2,2}^{1,1}\left[a|x|\middle|\begin{matrix}(a_1,A_1),(a_2,A_2)\\(b_1,B_1),(b_2,B_2)\end{matrix}\right], \tag{17}$$

similar to that for the case $\theta=0$, where $a=a(t)$ and N is the normalization constant. We want to determine the coefficients a_i, A_i, b_i, and B_i, as well as the function $a(t)$, by expansion of both sides of Eq. (13) in consecutive fractional powers of $|k|$ and neglect the terms which are small for $k\to 0$.

The function $p_\theta=x^{-\theta}p(x,t)$ on the right-hand side of Eq. (13) can be expressed as the Fox function due to the multiplication rule (A5). Its Fourier transform, given by Eq. (A6), expanded according to the formula (A3), takes the form

$$\tilde{p}_\theta(k,t)=2\pi[h_0^{(\theta)}+h_1^{(\theta)}|k'|+O(|k|^\mu)], \tag{18}$$

where $k'=Kk/a$. Deriving explicitly consecutive terms and utilizing simple properties of the gamma function, one can find conditions for the coefficients. First, to get the term of order k^0, we introduced the condition $1-a_1=A_1(1-\theta)$. $h_1^{(\theta)}=0$ identically because the gamma function in the denominator is infinite. Similarly, for the function $p(x,t)$ we have

$$\tilde{p}(k,t)=2\pi[h_0+h_{-\theta}|k'|^{-\theta}+h_1|k'|+h_2|k'|^2+h_\mu|k'|^\mu+O(|k|^{2\mu+\theta})], \tag{19}$$

where we imposed the condition $2-a_1=A_1(1+\mu)$ to get the exponent μ for the term $|k|^\mu$. Then the above conditions determine the first two coefficients of the Fox function: a_1 and A_1. We need also $h_{-\theta}=0$; this requirement can be satisfied if the argument of one of the gamma functions in the denominator assumes an integer and nonpositive value. The condition for that is $1-b_2-B_2(1-\theta)=0$ (alternatively, we could impose a similar condition for a_2 and A_2). The same procedure allows us to satisfy the requirement $h_2=0$ and to determine $B_2=1/(2+\theta)$. Finally, since the coefficients a_2 and A_2 enter the above expressions in a similar way as b_2 and B_2, we want to preserve this symmetry, present for the case $\theta=0$, and assume $a_2=b_2$ and $A_2=B_2$.

Then we insert the expansions Eqs. (18) and (19) into Eq. (13) and separate the time-dependent term: $(a/K)^{-\mu-\theta-1}\dot{a}/K=-\lambda$, where λ is a constant which scales the time and then it is not essential. We assume $\lambda=1/(\mu+\theta)$. By taking into account the initial condition for Eq. (14), one can write down the solution for the function $a(t)$ as

$$a=Kt^{-1/(\mu+\theta)} \tag{20}$$

and then reduce the problem to a simple equation:

$$-\frac{\mu}{\mu+\theta}h_\mu=h_0^{(\theta)\prime}, \tag{21}$$

where $h_0^{(\theta)\prime}=a^{-\theta}h_0^{(\theta)}$.

After implementing the coefficients we have evaluated, we obtain for the term h_μ the following expression: $h_\mu=\pi^{-2}(\mu+\theta)\Gamma(-\mu)\Gamma[b_1+B_1(1+\mu)]\cos(\mu\pi/2)\sin(\frac{\mu+\theta}{2+\theta}\pi)$. Unfortunately, the term $h_0^{(\theta)\prime}$ cannot be obtained directly from the series expansion because the undetermined expression emerges. Instead, we proceed as follows. $h_0^{(\theta)\prime}$ can be expressed as $h_0^{(\theta)\prime}=(2\pi)^{-1}a^{-\theta}\tilde{p}_\theta(k=0)=\pi^{-1}\int_0^\infty z^{-1}W(z)dz$, where we obtained the function

$$W(z)=H_{2,2}^{1,1}\left[z\middle|\begin{matrix}\left(1,\frac{1}{\mu+\theta}\right),\left(1,\frac{1}{2+\theta}\right)\\ [b_1+B_1(1-\theta),B_1],\left(1,\frac{1}{2+\theta}\right)\end{matrix}\right] \tag{22}$$

by applying the relation (A5). The required term can now be easily evaluated as the Mellin transform $\mathcal{M}(W)(s)=\int_0^\infty W(z)z^{s-1}dz$:

$$h_0^{(\theta)\prime}=\pi^{-1}\mathcal{M}(W)(s=0)=\pi^{-1}\chi(0)=\pi^{-1}\lim_{\epsilon\to 0}\chi(\epsilon)$$
$$=\frac{1}{\pi}\frac{\mu+\theta}{2+\theta}\Gamma[b_1+B_1(1-\theta)]. \tag{23}$$

Inserting the expressions for h_μ and $h_0^{(\theta)\prime}$ into Eq. (21) yields the relation between b_1 and B_1. Then, finally, the solution of Eq. (14) reads

$$p(x,t) = NaH_{2,2}^{1,1}\left[a|x| \left| \begin{matrix} \left(1-\frac{1-\theta}{\mu+\theta},\frac{1}{\mu+\theta}\right),\left(1-\frac{1-\theta}{2+\theta},\frac{1}{2+\theta}\right) \\ (b_1,B_1),\left(1-\frac{1-\theta}{2+\theta},\frac{1}{2+\theta}\right) \end{matrix} \right. \right] \tag{24}$$

with the condition

$$\frac{\mu}{\pi}\frac{2+\theta}{\mu+\theta}\Gamma(-\mu)\Gamma(b_1+B_1(1+\mu))\sin\left(\frac{\mu+\theta}{2+\theta}\pi\right)\cos(\mu\pi/2) + \Gamma[b_1+B_1(1-\theta)] = 0 \tag{25}$$

which follows directly from Eq. (21). The general theory of the Fox functions implies that $p(x,t)$ is the analytic function for all $x\neq 0$ if $B_1 > 1/(\mu+\theta)$. Apart from that—very weak—condition, one parameter is arbitrary. We will return to this issue in Sec. IV D. The normalization factor N can be determined in a simple way by using the Mellin transform: $N=[2\chi(-1)]^{-1}$, that yields

$$N = -\frac{\pi}{2}\left[\Gamma(b_1+B_1)\Gamma\left(-\frac{\theta}{\mu+\theta}\right)\sin\left(\frac{\theta}{2+\theta}\pi\right)\right]^{-1}. \tag{26}$$

In the k space, the solution (24) is of the form

$$\tilde{p}(k,t) = 1-\sigma^{\mu}|k|^{\mu}+\ldots \approx \exp(-\sigma^{\mu}|k|^{\mu}), \tag{27}$$

where

$$\sigma^{\mu} = \frac{K^{-\mu}}{\mu}\frac{(\mu+\theta)^2}{2+\theta}\frac{\Gamma(b_1+B_1(1-\theta))\Gamma\left(-\frac{\theta}{2+\theta}\right)}{\Gamma(b_1+B_1)\Gamma\left(-\frac{\theta}{\mu+\theta}\right)}t^{\mu/(\mu+\theta)} \tag{28}$$

and we neglected the terms of the order $|k|^{2\mu+\theta}$ ($\theta<0$) or $|k|^{2\mu}(\theta>0)$. The formula (27) means that within the scope of validity of our approximation the solution (24) is equivalent to the Lévy stable distribution.

The Fox functions can be evaluated from series expansions. Since they are poorly convergent, one needs the series both for small and large values of the argument. According to Eq. (A3), the expansion of $p(x,t)$ for small $|x|$ is of the form

$$p(x,t) = \frac{Na^{1+b_1/B_1}}{\pi B_1}|x|^{b_1/B_1}\sum_{i=0}^{\infty}\Gamma\left(\frac{1-\theta}{\mu+\theta}+\frac{1}{\mu+\theta}\frac{b_1+i}{B_1}\right) \times\sin\left[\left(\frac{1-\theta}{2+\theta}+\frac{1}{2+\theta}\frac{b_1+i}{B_1}\right)\pi\right](-1)^i(a|x|)^{i/B_1}/i!. \tag{29}$$

Using the property (A4), we obtain the series for large $|x|$:

$$p(x,t) = -\frac{Na}{\pi}(\mu+\theta)\sum_{i=1}^{\infty}\Gamma\{b_1+B_1[1-\theta+i(\mu+\theta)]\} \times\sin\left(\frac{\mu+\theta}{2+\theta}i\pi\right)(-1)^i(a|x|)^{-1+\theta-i(\mu+\theta)}/i!. \tag{30}$$

The above expression implies that the tail of the distribution has the same x dependence as for the case $\theta=0$: $p(x,t) \sim t^{\mu/(\mu+\theta)}/|x|^{1+\mu}$ ($|x|\to\infty$).

C. Diffusion

A characteristic feature of the Lévy distributions is the divergence of moments. In particular, the mean squared displacement is infinite for any time and then the transport phenomena require a modified formalism for the diffusion. One possibility is to substitute the variance by a moment of the order $\delta<\mu$.

In order to evaluate the moments of the distribution (24), we utilize properties of the Mellin transforms. A simple calculation yields

$$\begin{aligned}\langle|x|^{\delta}\rangle &= 2N\int_0^{\infty}x^{\delta}p(x,t)dx \\ &= 2Na^{-\delta}\chi(-\delta-1) \\ &= -\frac{2NK^{-\mu}}{\pi}\Gamma[b_1+B_1(1+\delta)]\Gamma\left(-\frac{\theta+\delta}{\mu+\theta}\right) \\ &\quad\times\sin\left(\frac{\theta+\delta}{2+\theta}\pi\right)t^{\delta/(\mu+\theta)}.\end{aligned} \tag{31}$$

Applying the above expression, one can compare individual cases in respect to the speed of transport. However, as long as the parameter δ is arbitrary, such formalism seems to be incomplete. Can it be fixed in some way? Clearly, the value $\delta=\mu$ is distinguished. Since that moment is divergent, we consider $\delta=\mu-\epsilon$, where $0<\epsilon\ll\theta$ and then the case $\theta=0$ is excluded. In the limit $\epsilon\to 0$, the gamma function can be expanded and Eq. (31) takes the form

$$\langle|x|^{\mu-\epsilon}\rangle \approx \frac{2NK^{-\mu}}{\pi\epsilon}(\mu+\theta)\Gamma[b_1+B_1(1+\mu)] \times\sin\left(\frac{\mu+\theta}{2+\theta}\pi\right)t^{\mu/(\mu+\theta)}. \tag{32}$$

Let us now define the fractional diffusion coefficient $\mathcal{D}^{(\mu)}(t)$:

$$\mathcal{D}^{(\mu)} \equiv \frac{1}{\Gamma(1+\mu)}\frac{1}{t}\lim_{\epsilon\to 0^+}\epsilon\langle|x|^{\mu-\epsilon}\rangle \quad (t\to\infty), \tag{33}$$

where $1<\mu<2$. According to Eq. (32), the limit is finite and $\mathcal{D}^{(\mu)} \sim t^{-\theta/(\mu+\theta)}$.

The interpretation of the above result is straightforward. If $\theta<0$ the coefficient $\mathcal{D}^{(\mu)}$ rises with time to infinity and we have the "superdiffusion." Conversely, for $\theta>0$ there is the "subdiffusion." Therefore we have obtained formally the same result as for FPE, Eq. (11), except the variance has

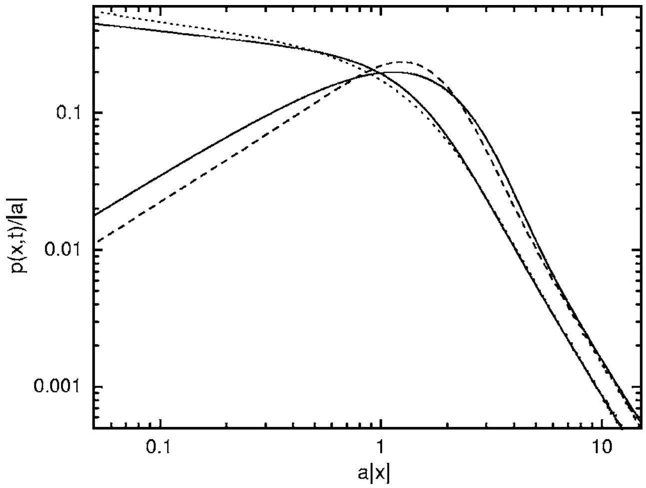

FIG. 2. The solutions of Eq. (2) for $\theta=-0.2$ (dots) and $\theta=1$ (dashed line). The corresponding solutions (24) are marked by solid lines.

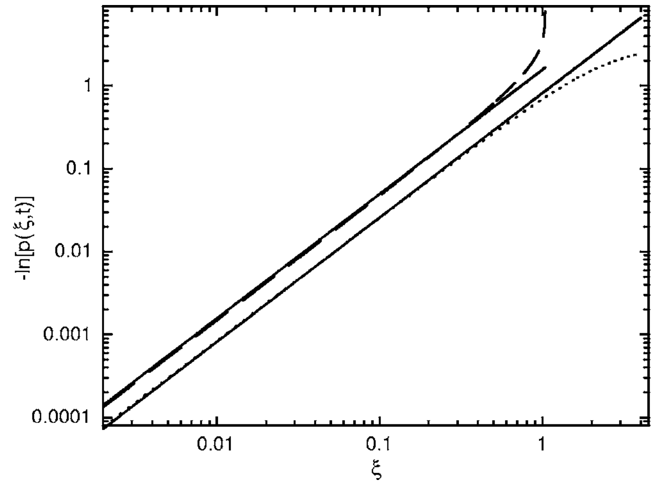

FIG. 3. Logarithm of the Fourier transform from the solution of Eq. (2) for $\theta=-0.2$ (dots) and $\theta=1$ (dashed line). The same quantity for the solution (24) which corresponds to both cases, calculated according to Eq. (27), is marked by solid lines; $\xi=k/a$.

been substituted by the fractional moment. The common conclusion, drawn both from the Gaussian and the fractional case, is that the sign of θ decides which kind of diffusion the system reveals.

D. Numerical examples

In this section we evaluate the probability distributions for specific cases. They are compared with the solutions of the master equation (2), obtained by the Monte Carlo simulations of stochastic trajectories of the jumping process. The Lévy-distributed jump-size density has been generated by using the algorithm from Ref. [44].

It follows from the expansion (29) that $p(x,t)\sim|x|^{b_1/B_1}$ for small $|x|$. Therefore the ratio of the coefficients b_1 and B_1 determines the shape of the distribution $p(x,t)$ there. Since our approximation is suited for large $|x|$, this ratio remains undetermined. The analysis of the master equation indicates that its solutions exhibit the power-law dependence of the form $|x|^\theta$ for small $|x|$. We utilize this property and assume the relation $b_1=\theta B_1$ in the present section; B_1 follows from the numerical solving of Eq. (25). The Fox functions have been evaluated from the series (29) and (30) with sufficient precision for all $|x|\in(0,\infty)$.

The solutions of the fractional equation (24), for a negative and a positive value of θ, are presented in Fig. 2. They correspond to the superdiffusion and subdiffusion, respectively, and display very different shapes. All the distributions with $\theta>0$ rise at small $|x|$ and display a maximum, whereas those for $\theta<0$ fall monotonically. The comparison with the solutions of the master equation is also presented in Fig. 2. The curves which represent both equations have similar shapes and their tails coincide.

Figure 3 presents the comparison of the Fourier transforms for the solution of both equations; the same cases as in Fig. 2 are shown: $\theta=-0.2$ and $\theta=1$. For that purpose, the master equation has been solved for all x up to very large values, then the numerical integration with the cosine function has been performed. In the case of the master equation solutions, the figure reveals substantial deviations from the straight lines—which represent the shape of the Lévy distribution—for large k: for $\theta<0$ that distribution stabilizes with k, whereas it falls rapidly to zero for $\theta>0$. At small k both solutions coincide with those of the fractional equation.

V. DISCUSSION

The jumping process we have discussed in this paper is Markovian because the waiting time probability distribution has the exponential form. However, since the jumping frequency depends on the process value, the system possesses some properties which are usually attributed to non-Markovian processes. In particular, we have demonstrated the presence of the anomalous diffusion.

If the step size is small, the jumping process can be regarded as continuous and described in terms of the FPE. That approximation has been accomplished by applying the Kramers-Moyal expansion, on the assumption that all moments of the step-size distribution are finite. It has been demonstrated—by solving the FPE exactly for the power-law frequency $\nu(x)$—that both normal and anomalous diffusion emerges.

On the other hand, we have considered the diffusion limit of the master equation for the step-size distribution which is stable and has divergent moments. The resulting equation (14) is fractional and contains a variable coefficient, in contrast to usually studied equations which govern Lévy processes. We have demonstrated that in the diffusion limit Eq. (14) is satisfied by the Fox function if ν depends algebraically on the position. The coefficients of the Fox function have been derived by inserting it to the equation and by comparing the terms. However, one parameter must remain undetermined if only the diffusion limit is considered because it is responsible for the behavior of the solution at small $|x|$.

The solution, since it is expressed as the Fox function, depends on time in the scaling form. Due to that property, simple conclusions about the transport can be drawn. The fractional moments are given by complicated expressions but the time dependence factorizes and it obeys a simple power law. We have defined the fractional diffusion coefficient $\mathcal{D}^{(\mu)}$ which allows us to establish a correspondence with the standard description in terms of the FPE. Both approaches predict the diffusion coefficient, either $\mathcal{D}$ or $\mathcal{D}^{(\mu)}$, in the form $\sim t^{-\theta/(\mu+\theta)}$, i.e., the subdiffusion ($\theta>0$) and the superdiffusion ($\theta<0$). We would like to emphasize that the above conclusions are independent of a specific choice of the free parameter in the solution.

An independent numerical analysis of the master equation (2) provides additional information about the jumping process. We can learn that it is not the Lévy stable process: substantial deviations from the Lévy distribution at large k are clearly visible. They become meaningless in the diffusion limit.

The comparison of the solutions of both diffusion equations we have discussed in this paper with the results of the numerical analysis of the master equation shows a reasonable agreement not only in the asymptotic limit of large $|x|$. Since the expressions (9) and (24) are relatively simple, compared to Eq. (3), both diffusion equations could serve as convenient approximations to the master equation (2).

APPENDIX

The Fox function [45–47] (for a review of the most important properties, see Refs. [21,42]) is defined as an inverse Mellin transform in the following way:

$$H_{pq}^{mn}\left[z\,\middle|\,\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] = H_{pq}^{mn}\left[z\,\middle|\,\begin{matrix}(a_1,A_1),(a_2,A_2),\ldots,(a_p,A_p)\\(b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)\end{matrix}\right] = \frac{1}{2\pi i}\int_L \chi(s)z^s ds, \tag{A1}$$

where

$$\chi(s) = \frac{\prod_1^m \Gamma(b_j - B_j s)\prod_1^n \Gamma(1-a_j+A_j s)}{\prod_{m+1}^q \Gamma(1-b_j+B_j s)\prod_{n+1}^p \Gamma(a_j - A_j s)}. \tag{A2}$$

Coefficients A_i and B_i are positive. The contour L is a straight line parallel to the imaginary axis which separates the poles of both gamma functions in Eq. (A2). If those poles are simple, the Fox function can be expressed in the form of the following series:

$$H_{pq}^{mn}\left[z\,\middle|\,\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] = \sum_{h=1}^{m}\sum_{\nu=0}^{\infty}\frac{\prod_{j=1,j\neq h}^{m}\Gamma\left(b_j - B_j\frac{b_h+\nu}{B_h}\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+A_j\frac{b_h+\nu}{B_h}\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+B_j\frac{b_h+\nu}{B_h}\right)\prod_{j=n+1}^{p}\Gamma\left(a_j - A_j\frac{b_h+\nu}{B_h}\right)}\frac{(-1)^\nu z^{(b_h+\nu)/B_h}}{\nu! B_h}. \tag{A3}$$

A similar expansion can be obtained for $z\to\infty$ by using the property

$$H_{pq}^{mn}\left[z\,\middle|\,\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] = H_{pq}^{mn}\left[\frac{1}{z}\,\middle|\,\begin{matrix}(1-b_q,B_q)\\(1-a_p,A_p)\end{matrix}\right]. \tag{A4}$$

Another useful property is the multiplication rule:

$$z^\sigma H_{pq}^{mn}\left[z\,\middle|\,\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] = H_{pq}^{mn}\left[z\,\middle|\,\begin{matrix}(a_p+\sigma A_p,A_p)\\(b_q+\sigma B_q,B_q)\end{matrix}\right]. \tag{A5}$$

The Fourier cosine transform of the Fox function yields

$$\int_0^\infty H_{pq}^{mn}\left[x\,\middle|\,\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]\cos(kx)dx = \frac{\pi}{k}H_{q+1,p+2}^{n+1,m}\left[k\,\middle|\,\begin{matrix}(1-b_q,B_q),(1,1/2)\\(1,1),(1-a_p,A_p),(1,1/2)\end{matrix}\right]. \tag{A6}$$

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